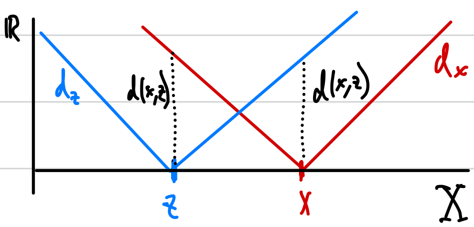
Math 451: Introduction to General Topology
Lecture 10

Difference form of triangle inequality. Let (X,d) (pseudo) metric space. Then for all x,y,z,
$$|d(x,y)-d(z,y)| \le d(x,z).$$

Proof. This is equivalent to $-d(x,z) \le d(x,y)-d(z,y) \le d(x,z)$ and both inequalities are just Δ-inequality: $d(x,y)-d(z,y) \le d(x,z) \iff d(x,y) \le d(x,z)+d(z,y)$; $-d(x,z) \le d(x,y)-d(z,y) \iff d(z,y) \le d(z,x)+d(x,y)$. $\square$

Theorem (Kaplinski). Every metric space (X,d) isometrically injects into $B(X,\mathbb{R})$. Thus, the closure of the image of X in $B(X,\mathbb{R})$ is a completion of X.

Proof. First consider, for each $x \in X$, the function $d_x : X \to \mathbb{R}$ by $d_x(y) := d(x,y)$.



Clearly, for any $x,z \in X$,
$$d_u(d_x, d_z) = d(x,z)$$
because of the triangle inequality (the difference form above) $|d(x,y)-d(z,y)| \le d(x,z)$ and the equality is achieved at $y=z$ and $y=x$. Thus, $x \mapsto d_x$ is an isometry wrt d_u on the right, but the d_x are not bounded in general, so the codomain isn't $B(X,\mathbb{R})$. — b for Busemann

To get bounded functions instead, we fix a point $x_0 \in X$ (think of it as a root or 0), and for each $x \in X$, define $b_x : X \to \mathbb{R}$ by $b_x := d_x - d_{x_0}$. Note that for any $y \in X$,
$$|b_x(y)| = |d_x(y) - d_{x_0}(y)| = |d(x,y) - d(x_0,y)| \underset{\text{diff } \Delta}{\le} d(x,x_0),$$
so $|b_x| \le d(x,x_0)$, i.e. b_x is bounded. And still, $x \mapsto b_x$ is an isometry:
$$d_u(b_x, b_z) = \sup_{y\in X} |b_x(y) - b_z(y)| = \sup_{y \in X} |d(x,y) - \cancel{d(x_0,y)} - d(z,y) + \cancel{d(x_0,y)}| =$$
$$= \sup_{y\in X} |d_x(y) - d_z(y)| = d_u(d_x, d_z) = d(x,z).$$

Thus, $x \mapsto b_x$ is an isometry $X \to B(X,\mathbb{R})$. Hence the closure $\overline{\{b_x : x \in X\}}$ is a completion of (X,d). $\boxed{\text{QED}}$

Continuity.

First we recall/define continuity for functions between metric spaces.

Def. For metric spaces (X, d_X), (Y, d_Y), and $x_0 \in X$, a function $f: X \to Y$ is said to be *continuous at x_0* if $\forall \varepsilon > 0\ \exists \delta > 0$ such that for all $x \in X$, whenever $d_X(x_0, x) < \delta$, we have $d_Y(f(x_0), f(x)) < \varepsilon$.
This is the same as $f(B_\delta(x_0)) \subseteq B_\varepsilon(f(x_0))$; equivalently, $B_\delta(x_0) \subseteq f^{-1}(B_\varepsilon(f(x_0)))$.

We rephrase this without mentioning ε, δ:

Prop. For metric spaces (X, d_X), (Y, d_Y), $x_0 \in X$, and a function $f: X \to Y$, TFAE:

(1) f is continuous at x_0, i.e. $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $B_\delta(x_0) \subseteq f^{-1}(B_\varepsilon(f(x_0)))$.

(1') for every open ball $B \subseteq Y$ centered at $f(x_0)$, $f^{-1}(B)$ contains an open ball centered at x_0.

(2) for every open set set $V \ni f(x_0)$, $\underbrace{f^{-1}(V) \text{ contains an open set } U \ni x_0}_{f^{-1}(V) \text{ is a } \textit{neighbourhood of } x_0}$.

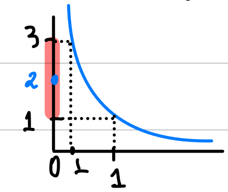
(2') the f-preimage of an open neighbourhood of $f(x_0)$ is a neighbourhood of x_0.

Proof. We only need to prove (1') $\Leftrightarrow$ (2) since the others are rephrasings.

(1') $\Rightarrow$ (2). Let $V \ni f(x_0)$ be open. Then $\exists \varepsilon > 0$ s.t. $B_\varepsilon(f(x_0)) \subseteq V$. By (1'), $\exists \delta > 0$ such that $B_\delta(x_0) \subseteq f^{-1}(B_\varepsilon(f(x_0))) \subseteq f^{-1}(V)$.

(2) $\Rightarrow$ (1'). Let $B := B_\varepsilon(f(x_0))$. By (2), $f^{-1}(B) \supseteq U \ni x_0$ where U is some open set. Thus $\exists \delta > 0$ s.t. $B_\delta(x_0) \subseteq U \subseteq f^{-1}(B_\varepsilon(f(x_0)))$. $\square$

Caution. In (2'), $f^{-1}(V)$ may not be open in general, even if V is open because of discontinuities at other points. For example, take $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) := \begin{cases} \frac{1}{x} & \text{if } x \neq 0 \\ 2 & \text{if } x = 0 \end{cases}$.
Then $f^{-1}((1,3)) = \{0\} \cup (\frac{1}{3}, 1)$, which isn't open.



Def. Let X, Y be metric spaces. A function $f: X \to Y$ is said to be *continuous* if it is continuous at every $x_0 \in X$.

Prop. For metric spaces X, Y, a function $f : X \to Y$ is continuous if and only if f-preimages of open sets are open, $f^{-1}(\text{open}) = \text{open}$.

Proof. $\Leftarrow$. Immediate from (2) above.

$\Rightarrow$. Let $V \subseteq Y$ be open. To show that $f^{-1}(V)$ is open, we fix $x_0 \in f^{-1}(V)$ and show that some ball centered at x_0 is $\subseteq f^{-1}(V)$. But this is true by (2) since $\exists$ open $x_0 \in U \subseteq f^{-1}(V)$, hence $\exists$ open ball B centered at x_0 with $B \subseteq U \subseteq f^{-1}(V)$. Thus, $f^{-1}(V)$ is open. $\square$

Remark. Topology is defined by open sets, so continuous functions are the homomorphism between the topologies.

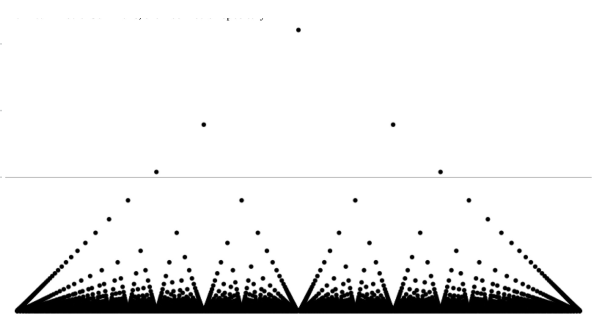
Example of continuous bijection with discontinuous inverse. Let $(X, d_X) := (\mathbb{R}, d_{01})$, where d_{01} is the 0/1-metric on $\mathbb{R}$. Let $(Y, d_Y) := (\mathbb{R}, \text{usual metric})$. Let $f : \mathbb{R} \to \mathbb{R}$ be the identity $x \mapsto x$. Then every subset of $\mathbb{R}$ is open wrt d_{01}-metric, so f is continuous. But $f^{-1} : (\mathbb{R}, \text{usual}) \to (\mathbb{R}, d_{01})$, and $\{0\}$ is open in d_{01}, but $f^{-1}(\{0\}) = \{0\}$ is not open in the usual metric of $\mathbb{R}$.

Examples. (a) Let $f : 10^{\mathbb{N}} \to [0,1]$ by $f(x) := 0.x_0x_1x_2\ldots$ treated as decimal representation of reals. This is surjective but not injective because some rationals have two different decimal representations, like $0.47999\ldots = 0.48000\ldots$ This map f is continuous because $f^{-1}(I)$, where $I \subseteq [0,1]$ is an intersection of an open interval with $[0,1]$ (e.g. $I = [0, \frac{1}{2}), (\frac{1}{3}, \frac{2}{3}), (\frac{1}{3}, 1]$) is open since if $x \in f^{-1}(I)$ then taking $n \in \mathbb{N}$ large enough, we get that $[x|_n] \subseteq f^{-1}(I)$ because if $0.x_0x_1\ldots x_{n-1}x_n\ldots$ and $0.y_0y_1\ldots y_{n-1}y_n\ldots$ coincide up to the n^{th} digit then their distance is $\leq 10^{-n}$.

We can do the same with $2^{\mathbb{N}} \to [0,1]$ by taking binary representations instead.

(b) **Thomae's function.** Let $T : (0,1) \to [0, \frac{1}{2}]$ be defined by

$$T(x) := \begin{cases} 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \\ \frac{1}{m} & \text{if } x \in \mathbb{Q} \text{ and } x = \frac{n}{m} \text{ is reduced.} \end{cases}$$



Claim. T is continuous at irrationals and discontinuous at rationals.

Proof. If $x = \frac{n}{m}$ is a rational and $\frac{n}{m}$ is reduced, then $T(x) = \frac{1}{m}$ and $V := (0, \frac{2}{m}) \ni \frac{1}{m}$ is open while $T^{-1}(V)$ only contains rationals, so it cannot contain an open interval $I \ni x$ because every nonempty open interval contains an irrational (HW).